

A CLASSICAL VERTEX ALGEBRA CONSTRUCTED WITH THE USE OF SOME LOGARITHMIC FORMAL CALCULUS

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ABSTRACT. Using some new logarithmic formal calculus, we construct a well known vertex algebra, obtaining the Jacobi identity directly, in an essentially self-contained treatment.

1. INTRODUCTION

Using some new logarithmic formal calculus, we construct a well known vertex algebra associated with $\widehat{\mathfrak{sl}(2)}$, obtaining the Jacobi identity directly, in an essentially self-contained treatment. This treatment is largely a (very) special case following the development of Chapter 8 and the relevant preliminaries in [FLM] (see also the elementary parts of [LL], [HLZ] and [M]). We introduce certain novelties, mostly having to do with some logarithmic calculus which grew out of considering a certain heuristic Remark 4.2.1 in [FLM]. The main idea is to systematically consider expressions like $e^{\log x}$, a formal power series in the formal variable $\log x$. Of course, $e^{\log x}$ is heuristically equivalent to x itself, and we suitably and carefully make this and similar identifications rigorous. The reader need not know anything about $\widehat{\mathfrak{sl}(2)}$, and indeed it is not even mentioned again in this paper. We only note it for those who already know the answer.

2. SOME PRELIMINARY FORMAL CALCULUS

We work in the standard formal calculus setting of [FLM] and [LL] and we start by briefly recalling some elementary facts treated in those works. We shall write x, y, z as well as $x_0, x_1, x_2, x_3, \dots, y_0, y_1, \dots, \xi_0, \xi_1, \dots$ etc. for commuting formal variables. In this paper, formal variables will always commute, and we will not use complex variables. All vector spaces will be over \mathbb{C} , although one may easily generalize many results to the case of a field of characteristic 0. Let V be a vector space. We use the following:

$$V[[x, x^{-1}]] = \left\{ \sum_{n \in \mathbb{Z}} v_n x^n \mid v_n \in V \right\}$$

(formal Laurent series), and some of its subspaces:

$$V((x)) = \left\{ \sum_{n \in \mathbb{Z}} v_n x^n \mid v_n \in V, v_n = 0 \text{ for sufficiently negative } n \right\}$$

(truncated formal Laurent series),

$$V[[x]] = \left\{ \sum_{n \geq 0} v_n x^n \mid v_n \in V \right\}$$

(formal power series),

$$V[x, x^{-1}] = \left\{ \sum_{n \in \mathbb{Z}} v_n x^n \mid v_n \in V, v_n = 0 \text{ for all but finitely many } n \right\}$$

(formal Laurent polynomials), and

$$V[x] = \left\{ \sum_{n \geq 0} v_n x^n \mid v_n \in V, v_n = 0 \text{ for all but finitely many } n \right\}$$

(formal polynomials). Often our vector space V will be a vector space of endomorphisms, $\text{End } V$. Even when V is replaced by $\text{End } V$ some of these spaces are not algebras, and we must define multiplication only up to a natural restrictive condition, a *summability condition*. Let $f_i(x) = \sum_{r \in \mathbb{C}} a_i(r) x^r \in (\text{End } V)\{x\}$ for $1 \leq i \leq m$. Then the product

$$f_1(x) f_2(x) \cdots f_k(x)$$

exists if for every $m \in \mathbb{C}$ and $v \in V$

$$\sum_{n_1 + \cdots + n_k = m} a_1(n_1) \cdots a_k(n_k) v$$

is a finite sum. We will use routine extensions of this principle of summability without further comment. One must be careful with existence issues when dealing with associativity properties of such products. For instance, if $F(x), G(x)$ and $H(x) \in \text{End } V[[x, x^{-1}]]$ and if the three products $F(x)G(x)$, $G(x)H(x)$ and $F(x)G(x)H(x)$ all exist, then it is easy to verify that

$$(F(x)G(x))H(x) = F(x)(G(x)H(x)),$$

but unless some such condition is checked then “associativity” may fail. We shall use this type of associativity and routine extensions of the principle without further comment.

Remark 2.1. Throughout this paper, we often extend our spaces to include more than one variable. We state certain properties which have natural extensions in such multivariable settings, which we will also use without further comment.

We shall frequently use the notation e^w to refer to the formal exponential expansion, where w is any formal object for which such expansion makes sense (meaning that the coefficients are finitely computable). For instance, we have the linear operator $e^{y \frac{d}{dx}} : \mathbb{C}[[x, x^{-1}]] \rightarrow \mathbb{C}[[x, x^{-1}]][[y]]$:

$$e^{y \frac{d}{dx}} = \sum_{n \geq 0} \frac{y^n}{n!} \left(\frac{d}{dx} \right)^n.$$

Similiarly we let $\log(1 + w)$ refer to the formal logarithmic series

$$\log(1 + w) = \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} w^i,$$

whenever the expansion makes sense. We note that we shall also be using objects with names like $\log x$ which is a symbol for a new atomic object, not a shorthand for a series, but rather a single new formal variable, and it should not be confused with $\log(1 + x)$ which is a series in the entirely different formal variable, x .

Proposition 2.1. *(The “automorphism property”) Let A be an algebra over \mathbb{C} . Let D be a formal derivation on A . That is, D is a linear map from A to itself which satisfies the product rule:*

$$D(ab) = (Da)b + a(Db), \text{ for all } a \text{ and } b \text{ in } A.$$

Then

$$e^{yD}(ab) = (e^{yD}a)(e^{yD}b),$$

Proof. Notice that

$$D^n ab = \sum_{l+k=n}^r n! \frac{D^k a}{k!} \frac{D^l b}{l!}.$$

Then divide both sides by $n!$ and sum over y and the result follows. \square

Essentially for the same reason as the above proof (combinatorially speaking) we have for w_1 and w_2 commuting objects, the formal rule

$$(2.1) \quad e^{w_1+w_2} = e^{w_1}e^{w_2}.$$

More generally, we shall find the following formal rule very useful. If $[x, y]$ commutes with x and y then

$$(2.2) \quad e^x e^y = e^y e^x e^{[x,y]}.$$

We essentially follow the exposition for formula (3.4.7) in [FLM].

$$xe^y = x \sum_{n \geq 0} \frac{y^n}{n!} = \sum_{n \geq 0} \frac{y^n}{n!} x + \sum_{n \geq 1} \frac{ny^{n-1}}{n!} [x, y] = e^y x + e^y [x, y] = e^y (x + [x, y]),$$

and iterating gives

$$x^k e^y = e^y (x + [x, y])^k,$$

and dividing by $k!$ and summing over $k \geq 0$ gives

$$e^x e^y = e^y e^{x+[x,y]},$$

which gives the result because $[x, y]$ commutes with x .

We define

$$(2.3) \quad \binom{n}{k} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!}$$

for all $n \in \mathbb{C}$ and $k \in \mathbb{Z}$.

Definition 2.1.

$$(x+y)^r = e^{y \frac{d}{dx}} x^r \quad \text{for } r \in \mathbb{C}.$$

This definition is equivalent to the usual *binomial expansion convention*, the convention being that the second listed variable is expanded in nonnegative powers. The reader may easily check that this definition coincides with the usual definition of $(x+y)^n$ when n is a nonnegative integer. Other expansions, using (2.3), also “look” as expected. But we must be careful. The following nontrivial fact is true.

Proposition 2.2. *For all $n \in \mathbb{Z}$,*

$$(x + (y+z))^n = ((x+y) + z)^n.$$

Proof. Recalling (2.1), we have

$$(x + (y+z))^n = e^{(y+z) \frac{\partial}{\partial x}} x^n = e^{y \frac{\partial}{\partial x}} \left(e^{z \frac{\partial}{\partial x}} x^n \right) = e^{y \frac{\partial}{\partial x}} (x+z)^n = ((x+y) + z)^n.$$

□

Given formal commuting variables x, y etc., we take the point of view of certain elementary material in [M] and [HLZ] and let $\log x, \log y$ etc. be formal variables commuting with x, y and each other etc. We further require that $\frac{d}{dx} \log x = x^{-1}$ and extend the operator to act as the unique possible derivation on $\mathbb{C}[x, \log x]$. We further “formally linearly complete” $\frac{d}{dx}$ to act on $\mathbb{C}[[x, x^{-1}, \log x, (\log x)^{-1}]]$ in the obvious way. We shall use other similar routine definitions without further comment.

Definition 2.2.

$$(\log(x+y))^r = e^{y \frac{d}{dx}} (\log x)^r \quad r \in \mathbb{C}.$$

It is easy to verify that

$$\log(x+y) = \log x + \log \left(1 + \frac{y}{x} \right).$$

The following is a special case of the first part of Theorem 3.6 in [HLZ].

Proposition 2.3. *(The logarithmic formal Taylor theorem)*

For $p(x) \in \mathbb{V}[[x, x^{-1}, \log x, (\log x)^{-1}]]$,

$$e^{y \frac{d}{dx}} p(x) = p(x+y).$$

Proof. A summand of $p(x)$ is doubly weighted by the degrees of the formal variables. The summands of $p(x + y)$ are doubly weighted by the total degree in x and y as well as separately in the degree in $\log x$. The operator preserves the total degree of the x and y variables. Thus we only need to consider $p(x)$ of the form $x^k q(x)$ where $k \in \mathbb{Z}$ and $q(x) \in \mathbb{V}[[\log x, (\log x)^{-1}]]$. Let $q_n(x)$ be the Laurent polynomial which is equal to $q(x)$ but truncated in powers whose absolute value is less than n . If we focus on a fixed power of y and separately of $\log x$ it is clear that we only need to consider $q_n(x)$ in place of $q(x)$. The result follows by linearity after considering the trivial cases $p(x) = x^r$ and $p(x) = (\log x)^r$ for $r \in \mathbb{C}$ and applying the automorphism property. \square

Proposition 2.4. *The vectors e^{mx} for $m \in \mathbb{C}$ are linearly independent.*

Proof. Otherwise there is a linear combination $\sum_{m \in \mathbb{C}} a_m e^{mx} = 0$, where $a_m \in \mathbb{C}$ and only finitely many coefficients are nonzero. Considering the higher derivatives and focusing on the constant term yields a Vandermonde matrix which is nonsingular which gives that the coefficients must all be zero. \square

We have therefore that the space spanned by the vectors $e^{m\xi_0}$ $m \in \mathbb{Z}$ is the space of Laurent polynomials $\mathbb{C}[e^{\xi_0}, e^{-\xi_0}]$. We tensor this space with the space of polynomials $\mathbb{C}[\xi_1, \xi_2, \dots]$ to get the space

$$(2.4) \quad \Xi = \mathbb{C}[e^{\xi_0}, e^{-\xi_0}, \xi_1, \xi_2, \dots].$$

Our vertex algebra will have Ξ as its underlying space. Of course, Ξ is also an algebra under the obvious rules.

3. THE DELTA FUNCTION

We define the formal delta function by

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n.$$

Certain elementary identities concerning delta functions are very convenient for dealing with the arithmetic of vertex algebras and, in fact, in some cases, are fundamental to the very notion of vertex algebra. We state and prove some such identities in this section. The following identity appeared in [FLM]. Our proof follows that given in [R].

Proposition 3.1. *We have the following two elementary identities:*

$$(3.1) \quad y^{-1} \delta \left(\frac{x - z}{y} \right) - x^{-1} \delta \left(\frac{y + z}{x} \right) = 0$$

and

$$(3.2) \quad z^{-1} \delta \left(\frac{x - y}{z} \right) - z^{-1} \delta \left(\frac{-y + x}{z} \right) - x^{-1} \delta \left(\frac{y + z}{x} \right) = 0.$$

Proof. First observe that

$$(3.3) \quad y^{-1}\delta\left(\frac{x}{y}\right) = \sum_{l<0} x^l y^{-l-1} + \sum_{l\geq 0} x^l y^{-l-1} = e^{-y\frac{d}{dx}} x^{-1} + e^{-x\frac{d}{dy}} y^{-1} = (x-y)^{-1} + (y-x)^{-1}.$$

Then by the formal Taylor theorem

$$\begin{aligned} y^{-1}\delta\left(\frac{x+z}{y}\right) &= e^{z\frac{d}{dx}} y^{-1}\delta\left(\frac{x}{y}\right) \\ &= e^{z\frac{d}{dx}} ((x-y)^{-1} + (y-x)^{-1}) \\ &= ((x+z)-y)^{-1} + (y-(x+z))^{-1}. \end{aligned}$$

Being careful with minus signs, we may respectively expand all the terms in the left-hand side of (3.2) to get

$$\begin{aligned} &((x-y)-z)^{-1} + (z-(x-y))^{-1} \\ &-((-y+x)-z)^{-1} - (z-(-y+x))^{-1} \\ &-((y+z)-x)^{-1} - (x-(y+z))^{-1}. \end{aligned}$$

Now we get by Proposition 2.2 that the first and sixth terms, the third and fifth terms, and the second and fourth terms pairwise cancel each other thus giving us (3.2). The other identity may be proved in a similar fashion. \square

We note the following easily verified identity extending (3.3):

$$(3.4) \quad \frac{1}{n!} \left(\frac{\partial}{\partial y}\right)^n y^{-1}\delta\left(\frac{x}{y}\right) = \frac{(-1)^n}{n!} \left(\frac{\partial}{\partial x}\right)^n y^{-1}\delta\left(\frac{x}{y}\right) = (x-y)^{-n-1} - (-y+x)^{-n-1}.$$

In addition to the above identities, delta functions have certain crucial substitution properties. To state such a property in the generality needed, we need the following logarithmic substitution operator. It is crucial to note as in [HLZ] that whereas

$$\log(e^x) = \log(1 + (e^x - 1)) = x$$

it is *not* true that

$$e^{\log x} = x,$$

since the left hand side is a series in the variable $\log x$. But nonetheless, the two expressions are heuristically equal and we develop a means of rigorizing this heuristic suggestion next. For V which does not depend on $\log x$ or x we shall define an operator

$$\phi_x : V[[x, x^{-1}]] [e^{\log x}, e^{-\log x}] \rightarrow V[[x, x^{-1}]].$$

The operator ϕ_x essentially substitutes x for $e^{\log x}$. For $f(x) = \sum_{n \in \mathbb{Z}} v_n(x) e^{n \log x} \in V[[x, x^{-1}]] [e^{\log x}, e^{-\log x}]$ where $v_n(x) \in V[[x, x^{-1}]]$, let

$$\phi_x(f(x)) = \sum_{n \geq 0} v_n(x) x^n.$$

This is well defined because $v_n(x) x^n$ is a well defined formal Laurent series and $\phi_x(f(x))$ is a finite sum of such series. We shall use the notation

$$\phi_{x,y} = \phi_x \circ \phi_y$$

etc., if we wish to indicate the operator substituting for more than one logarithmic variable in a similar way.

Proposition 3.2. *The operator ϕ_x commutes with $\frac{d}{dx}$.*

Proof. It is enough to consider acting on elements of the form $x^n e^{m \log x}$. We have

$$\phi_x \frac{d}{dx} x^n e^{m \log x} = \phi_x ((n+m)x^{n-1} e^{m \log x}) = (m+n)x^{m+n-1}$$

and

$$\frac{d}{dx} \phi_x x^n e^{m \log x} = \frac{d}{dx} x^{n+m} = (m+n)x^{m+n-1}.$$

□

The following is an example of how we shall typically be applying ϕ_x ,

$$\phi_x e^{-2 \log(x-z)} = \phi_x e^{-z \frac{\partial}{\partial x}} e^{-2 \log x} = e^{-z \frac{\partial}{\partial x}} \phi_x e^{-2 \log x} = e^{-z \frac{\partial}{\partial x}} x^{-2} = (x-z)^{-2}.$$

Proposition 3.3. *For*

$$f(x, y, z) \in \text{End } V[[x, x^{-1}, e^{\log x}, e^{-\log x}, y, y^{-1}, e^{\log y}, e^{-\log y}, z, z^{-1}, e^{\log z}, e^{-\log z}]]$$

such that for each fixed $v \in V$

$$f(x, y, z)v \in \text{End } V[[x, x^{-1}, y, y^{-1}]] [e^{\log x}, e^{-\log x}, e^{\log y}, e^{-\log y}]((z)) [e^{\log z}, e^{-\log z}]$$

and such that

$$\lim_{x \rightarrow y} f(x, y, z)$$

exists (where the “limit” is the indicated formal substitution), we have

$$\begin{aligned} \phi_{x,y,z} \delta \left(\frac{y+z}{x} \right) f(x, y, z) &= \phi_{x,y,z} \delta \left(\frac{y+z}{x} \right) f(y+z, y, z) \\ &= \phi_{x,y,z} \delta \left(\frac{y+z}{x} \right) f(x, x-z, z). \end{aligned}$$

Proof. Since we are substituting terms homogeneous of degree 1, for other terms homogeneous of degree 1, therefore after $\phi_{x,y,z}$ is applied we need only consider homogeneous $f(x, y, z)$. The limit restriction on $f(x, y, z)$ together with the fact that the logarithmic terms only involve finite sums means that we need only consider when

$$f(x, y, z) = x^{l_1} e^{l_2 \log x} y^{m_1} e^{m_2 \log y} z^{n_1} e^{n_2 \log z}.$$

This follows from an easy calculation. \square

4. SOME GENERAL LOG VERTEX OPERATORS

Let $x, \log x$ and ξ_i for $i \geq 0$ be formal commuting variables and consider the space $\mathbb{C}[\xi_1, \xi_2, \dots][[\xi_0]]$. We define some operators on this space (cf. (4.2.1) in [FLM]). Let

$$h^+(x) = 2 \log x \frac{\partial}{\partial \xi_0} - 2 \sum_{n>0} \frac{x^{-n}}{n} \frac{\partial}{\partial \xi_n}$$

and

$$h^-(x) = \sum_{m \geq 0} \xi_m x^m.$$

Let

$$h(x) = h^+(x) + h^-(x),$$

so that

$$h(x) : \mathbb{C}[\xi_1, \xi_2, \dots][[\xi_0]] \rightarrow \mathbb{C}[\xi_1, \xi_2, \dots][[\xi_0]]((x))[\log x].$$

We use colons to denote normally ordered products, where normal ordering places differential operators to the right side of products. For instance, for $n \geq 0$

$$: \frac{\partial}{\partial \xi_n} \xi_n := \xi_n \frac{\partial}{\partial \xi_n} := \xi_n \frac{\partial}{\partial \xi_n},$$

with the notation extended in the obvious way over longer products. We have

$$: h^-(x) h^+(x) := h^-(x) h^+(x) := h^-(x) h^+(x),$$

so that for $m \in \mathbb{Z}$ and $n \geq 0$

$$(4.5) \quad : mh(x)^n := (mh^-(x) + mh^+(x))^n := \sum_{l+k=n} \frac{(mh^-(x))^l}{l!} \frac{(mh^+(x))^k}{k!}.$$

Let us consider $: mh(x)^n :$ acting on an element of $\mathbb{C}[\xi_1, \xi_2, \dots][[\xi_0]]$. In particular, focusing on the coefficient of x and separately $\log x$ and separately x_0 all raised to fixed powers then it is clear that the answer is zero for sufficiently large n . Therefore dividing by $n!$ and summing, we get the well defined operator (cf. (4.2.6) and also (4.2.12) in [FLM])

$$: e^{mh(x)} :.$$

Recall (2.4), the space

$$\Xi = \mathbb{C}[e^{\xi_0}, e^{-\xi_0}, \xi_1, \xi_2, \dots].$$

We may consider $: e^{mh(x)} :$ acting on Ξ as in the next proposition.

Proposition 4.1. *For $m \in \mathbb{Z} : e^{mh(x)} :: \Xi \rightarrow \Xi((x))[e^{\log x}, e^{-\log x}]$*

$$: e^{mh(x)} := e^{mh^-(x)} e^{mh^+(x)} = e^{mx_0} e^{mh^-(x)-mx_0} e^{mh^+(x)-2\frac{\partial}{\partial \xi_0} \log x} e^{2\frac{\partial}{\partial \xi_0} \log x}.$$

Proof. Divide (4.5) by $n!$ and sum over $n \geq 0$. That we may restrict the domain and range follows easily from the expanded formula. \square

We define a linear and “normal multiplicative” map on Ξ . We shall explain “normal multiplicative” in a moment. First let

$$(4.6) \quad \Upsilon(1, x) = 1.$$

Next, for $n \geq 1$, let

$$\Upsilon(\xi_n, x) = \frac{h^{(n)}(x)}{n!}.$$

And extend naturally to let

$$(4.7) \quad \Upsilon(e^{m\xi_0}, x) =: e^{mh(x)} :,$$

for $m \in \mathbb{Z}$.

Remark 4.1. The reader should compare the operator (4.7) with the heuristic operator in Remark 4.2.1 of [FLM], which provided the original motivation for developing this, now rigorous, operator.

Then for $p, q \in \Xi$ we require that

$$(4.8) \quad \Upsilon(pq, x) =: \Upsilon(p, x)\Upsilon(q, x) :,$$

(which is what we mean by “normal multiplicative”). This determines a unique linear map

$$\Upsilon(\cdot, x) : \Xi \otimes \Xi \rightarrow \Xi((x))[e^{\log x}, e^{-\log x}].$$

Extending the notation in the obvious way, we have

$$\Upsilon(h^-(z), x) = \Upsilon\left(\sum_{m \geq 0} \xi_m z^m, x\right) = \sum_{m \geq 0} \frac{h^{(m)}(x)}{m!} z^m = e^{z \frac{d}{dx}} h(x) = h(x + z).$$

So

$$(4.9) \quad \Upsilon(e^{h^-(z)}, x) =: e^{h(x+z)} :,$$

and the range of this operator is $\Xi((x))[e^{\log x}, e^{-\log x}][[z]]$, which is easy to see using Proposition 2.3. More generally, the range of

$$: e^{h(x+y)} e^{h(x+z)} :$$

is $\Xi((x))[e^{\log x}, e^{-\log x}][[y, z]]$ which can be seen by first considering

$$: e^{h(x_1+y)} e^{h(x_2+z)} :$$

and then setting x_1 and x_2 equal to x . We shall use such reasoning without comment below.

We have

$$\begin{aligned} h^+(x) e^{nh^-(y)} &= \left(2 \log x \frac{\partial}{\partial \xi_0} - 2 \sum_{k>0} \frac{x^{-k}}{k} \frac{\partial}{\partial \xi_k} \right) e^{n \sum_{m \geq 0} \xi_m y^m} \\ &= \left(2n \log x - 2 \sum_{k>0} \frac{n}{k} y^k x^{-k} \right) e^{nh^-(y)} \\ &= 2n \left(\log x + \log \left(1 - \frac{y}{x} \right) \right) e^{nh^-(y)} \\ &= 2n \log(x-y) e^{nh^-(y)}, \end{aligned}$$

so that, letting $k = 2mn$,

$$(4.10) \quad e^{-mh^+(x)} e^{nh^-(y)} = e^{-k \log(x-y)} e^{nh^-(y)}.$$

We note that we may set $y = 0$ in this formula. We will use this later.

5. JACOBI IDENTITY

For $m, n \in \mathbb{Z}$, recalling (2.2)

$$: e^{mh(x)} :: e^{nh(z)} : = e^{mh^-(x)} e^{mh^+(x)} e^{nh^-(z)} e^{nh^+(z)} =: e^{mh(x)+nh(z)} : e^{[mh^+(x), nh^-(z)]},$$

because (cf. the proof of Proposition 4.3.1 in [FLM])

$$\begin{aligned} [h^+(x), h^-(z)] &= \left[2 \frac{\partial}{\partial \xi_0} \log x - 2 \sum_{n>0} \frac{1}{n} \frac{\partial}{\partial \xi_n} x^{-n}, \sum_{m \geq 0} \xi_m z^m \right] \\ &= \left[2 \frac{\partial}{\partial \xi_0}, \xi_0 \right] \log x + \sum_{n, m > 0} \left[\frac{-2}{n} \frac{\partial}{\partial \xi_n}, \xi_m \right] x^{-n} z^m \\ &= 2 \log x - 2 \sum_{n>0} \frac{1}{n} \left(\frac{z}{x} \right)^n \\ &= 2 \log x + 2 \log \left(1 - \frac{z}{x} \right) \\ &= 2 \log(x-z) \end{aligned}$$

is central, and moreover we have

$$(5.11) \quad : e^{mh(x)} :: e^{nh(z)} : =: e^{mh(x)+nh(z)} : e^{2mn \log(x-z)}.$$

The reader should compare the calculation here with the work in Section 8.6 of [FLM]. Let x_i, y_i, z_i for $i \geq 1$ and for convenience we also let $x_0 = y_0 = z_0 = 0$. Then let

$$A = e^{\sum_{i \geq 1} -m_i h^-(x_i)} e^{-m_0 \xi_0}$$

and

$$B = e^{\sum_{j \geq 1} n_j h^-(y_j)} e^{n_0 \xi_0},$$

where $m_i, n_j \in \mathbb{Z}$ and only finitely many are nonzero (cf. (8.6.5) in [FLM]). Then letting $k_{ij} = 2m_i n_j$,

$$\begin{aligned} \phi_{x,y,z} \Upsilon(A, x) \Upsilon(B, y) &= \phi_{x,y,z} : e^{\sum_{i \geq 0} -m_i h(x+x_i)} :: e^{\sum_{j \geq 0} n_j h(y+y_j)} : \\ &= \phi_{x,y,z} : e^{\sum_{i \geq 0} -m_i h(x+x_i) + \sum_{j \geq 0} n_j h(y+y_j)} : e^{\sum_{i,j \geq 0} -k_{ij} \log(x+x_i-y-y_j)} \\ &= \phi_{x,y,z} : e^{\sum_{i \geq 0} -m_i h(x+x_i) + \sum_{j \geq 0} n_j h(y+y_j)} : \prod_{i,j \geq 0} (x+x_i-y-y_j)^{-k_{ij}}, \end{aligned}$$

by (4.8), (4.9) and (5.11) and its range is $\Xi((x))[[x_1, x_2, \dots]]((y))[[y_1, y_2, \dots]]$. Therefore we may multiply this expression by $z^{-1} \delta\left(\frac{x-y}{z}\right)$ and Proposition 3.3 may be applied to give

$$\begin{aligned} \phi_{x,y,z} z^{-1} \delta\left(\frac{x-y}{z}\right) \Upsilon(A, x) \Upsilon(B, y) \\ (5.12) \quad = \phi_{x,y,z} z^{-1} \delta\left(\frac{x-y}{z}\right) : e^{\sum_{i \geq 0} -m_i h(x+x_i) + \sum_{j \geq 0} n_j h(y+y_j)} : \prod_{i,j \geq 0} (z+x_i-y_j)^{-k_{ij}}. \end{aligned}$$

Similarly

$$\begin{aligned} \phi_{x,y,z} z^{-1} \delta\left(\frac{y-x}{-z}\right) \Upsilon(B, y) \Upsilon(A, x) \\ (5.13) \quad = \phi_{x,y,z} z^{-1} \delta\left(\frac{y-x}{-z}\right) : e^{\sum_{i \geq 0} -m_i h(x+x_i) + \sum_{j \geq 0} n_j h(y+y_j)} : \prod_{i,j \geq 0} (z+x_i-y_j)^{-k_{ij}}, \end{aligned}$$

noting that $(-z-x_i+y_j)^{-k_{ij}} = (z+x_i-y_j)^{-k_{ij}}$, since k_{ij} is always even.

Further

$$\begin{aligned} \Upsilon(\Upsilon(A, z) B, y) &= \Upsilon\left(: e^{\sum_{i \geq 0} -m_i h(z+x_i)} : e^{\sum_{j \geq 1} n_j h^-(y_j) + n_0 \xi_0}, y\right) \\ &= e^{\sum_{i,j \geq 0} -k_{ij} \log(z+x_i-y_j)} \Upsilon\left(e^{\sum_{i \geq 0} -m_i h^-(z+x_i) + \sum_{j \geq 1} n_j h^-(y_j) + n_0 \xi_0}, y\right) \\ &= e^{\sum_{i,j \geq 0} -k_{ij} \log(z+x_i-y_j)} : e^{\sum_{i \geq 0} -m_i h(y+z+x_i) + \sum_{j \geq 0} n_j h(y+y_j)} :, \end{aligned}$$

by (4.8), (4.9) and (4.10), and its range is

$$\Xi((y))[e^{\log y}, e^{-\log y}][[y_1, y_2, \dots]]((z))[e^{\log z}, e^{-\log z}][[x_1, x_2, \dots]],$$

so that we may multiply by $y^{-1}\delta\left(\frac{x-z}{y}\right)$ and Proposition 3.3 applies, giving

$$\begin{aligned}
 & \phi_{x,y,z} x^{-1} \delta\left(\frac{y+z}{x}\right) \Upsilon(\Upsilon(A, z) B, y) \\
 (5.14) \quad &= \phi_{x,y,z} x^{-1} \delta\left(\frac{y+z}{x}\right) \prod_{i,j \geq 0} (z + x_i - y_j)^{-k_{ij}} : e^{\sum_{i \geq 0} -m_i h(x+x_i) + \sum_{j \geq 0} n_j h(y+y_j)} : .
 \end{aligned}$$

Now adding the left hand sides of (5.12), (5.13) and (5.14) together, factoring and using Proposition 3.1, we get:

$$\begin{aligned}
 & \phi_{x,y,z} z^{-1} \delta\left(\frac{x-y}{z}\right) \Upsilon(A, x) \Upsilon(B, y) - \phi_{x,y,z} z^{-1} \delta\left(\frac{y-x}{-z}\right) \Upsilon(B, y) \Upsilon(A, x) \\
 (5.15) \quad &= \phi_{x,y,z} x^{-1} \delta\left(\frac{y+z}{x}\right) \Upsilon(\Upsilon(A, z) B, y) .
 \end{aligned}$$

6. A VERTEX ALGEBRA

The operator $\Upsilon(\cdot, x)$ is almost a vertex operator, but we need to specialize $e^{\log x}$ to x . So we let

$$(6.16) \quad Y(\cdot, x) \cdot = \phi_x \circ \Upsilon(\cdot, x) \cdot$$

so that

$$Y(\cdot, x) \cdot : \Xi \otimes \Xi \rightarrow \Xi((x))$$

is a linear map.

We are now ready to recall the definition of a vertex algebra. Individual mathematical vertex operators were introduced in [LW]. The notion of vertex algebra was first mathematically defined by Borchers in [B]. An equivalent set of axioms, based primarily on the Jacobi identity, which we shall use, appeared first in [FLM] as part of the notion of vertex operator algebra (cf. [LL], in which the equivalence of the two sets of axioms is proved).

Definition 6.1. A *vertex algebra* is a vector space equipped, first, with a linear map (the *vertex operator map*) $V \otimes V \rightarrow V[[x, x^{-1}]]$, or equivalently, a linear map

$$\begin{aligned}
 Y(\cdot, x) : V &\rightarrow (\text{End} V)[[x, x^{-1}]] \\
 v &\mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} .
 \end{aligned}$$

We call $Y(v, x)$ the *vertex operator associated with* v . We assume that

$$Y(u, x)v \in V((x))$$

for all $u, v \in V$. There is also a distinguished element $\mathbf{1}$ satisfying the following *vacuum property* :

$$Y(\mathbf{1}, x) = 1$$

and *creation property* :

$$\begin{aligned} Y(u, x)\mathbf{1} &\in V[[x]] \text{ and} \\ Y(u, 0)\mathbf{1} &= u \text{ for all } u \in V. \end{aligned}$$

Finally, we require that the *Jacobi identity* is satisfied:

$$\begin{aligned} z^{-1}\delta\left(\frac{x-y}{z}\right)Y(u, x)Y(v, y) - z^{-1}\delta\left(\frac{y-x}{-z}\right)Y(v, y)Y(u, x) \\ = x^{-1}\delta\left(\frac{y+z}{x}\right)Y(Y(u, z)v, y). \end{aligned}$$

Theorem 6.1. *The space Ξ equipped with $Y(\cdot, x)\cdot$ as defined by (6.16), is a vertex algebra, with vacuum vector $\mathbf{1}$.*

Proof. The minor conditions, such as truncation have already been dealt with. The vacuum property follows by (4.6). To see the creation property, note that

$$Y(A, x)\mathbf{1} = \phi_x : e^{\sum_{i \geq 0} -m_i h(x+x_i)} : \mathbf{1} = e^{\sum_{i \geq 0} -m_i h^-(x+x_i)},$$

which is a power series in x and we may substitute $x = 0$ which is easily seen to give

$$Y(A, 0)\mathbf{1} = A.$$

Noting (5.15), we will be done if we can show that the coefficients in A and B span Ξ as we let their parameters vary.

The following is essentially the same observation as in Remark 8.3.9 in [FLM]. Let

$$e^{\sum_{n \geq 1} \xi_n y^n} = \sum_{k \geq 0} p_k y^k \quad \text{so that} \quad \sum_{n \geq 1} \xi_n y^n = \log \left(1 + \sum_{k \geq 1} p_k y^k \right),$$

which shows that any polynomial in the ξ_i $i \geq 1$ is a polynomial in the p_i for $i \geq 1$. This gives the result. \square

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A CLASSICAL VERTEX ALGEBRA CONSTRUCTED WITH THE USE OF SOME LOGARITHMIC FORMAL CALCULUS

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ABSTRACT. Using some new logarithmic formal calculus, we construct a well known vertex algebra, obtaining the Jacobi identity directly, in an essentially self-contained treatment.

1. INTRODUCTION

Using some new logarithmic formal calculus, we construct a well known vertex algebra associated with $\widehat{\mathfrak{sl}(2)}$, obtaining the Jacobi identity directly, in an essentially self-contained treatment. This treatment is largely a (very) special case following the development of Chapter 8 and the relevant preliminaries in [FLM] (see also the elementary parts of [LL], [HLZ] and [M1]). We introduce certain novelties, mostly having to do with some logarithmic calculus which grew out of considering a certain heuristic Remark 4.2.1 in [FLM]. The main idea is to systematically consider expressions like $e^{\log x}$, a formal power series in the formal variable $\log x$. Of course, $e^{\log x}$ is heuristically equivalent to x itself, and we suitably and carefully make this and similar identifications rigorous. The reader need not know anything about $\widehat{\mathfrak{sl}(2)}$, and indeed it is not even mentioned again in this paper. We only note it for those who already know the answer.

We wish to direct the reader to an independent use of similar formal logarithmic objects as those considered in this paper. In [M2] (see especially Section 8), the author rigorously uses certain exponentiated logarithmic terms similar to those considered here in order to construct certain operators which they call “mock logarithmic intertwining operators.”

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2. SOME PRELIMINARY FORMAL CALCULUS

We work in the standard formal calculus setting of [FLM] and [LL] and we start by briefly recalling some elementary facts treated in those works. We shall write x, y, z as well as $x_0, x_1, x_2, x_3, \dots, y_0, y_1, \dots, \xi_0, \xi_1, \dots$ etc. for commuting formal variables. In this paper, formal variables will always commute, and we will not use complex variables. All vector spaces will be over \mathbb{C} , although one may easily generalize many results to the case

of a field of characteristic 0. Let V be a vector space. We use the following:

$$V[[x, x^{-1}]] = \left\{ \sum_{n \in \mathbb{Z}} v_n x^n \mid v_n \in V \right\}$$

(formal Laurent series), and some of its subspaces:

$$V((x)) = \left\{ \sum_{n \in \mathbb{Z}} v_n x^n \mid v_n \in V, v_n = 0 \text{ for sufficiently negative } n \right\}$$

(truncated formal Laurent series),

$$V[[x]] = \left\{ \sum_{n \geq 0} v_n x^n \mid v_n \in V \right\}$$

(formal power series),

$$V[x, x^{-1}] = \left\{ \sum_{n \in \mathbb{Z}} v_n x^n \mid v_n \in V, v_n = 0 \text{ for all but finitely many } n \right\}$$

(formal Laurent polynomials), and

$$V[x] = \left\{ \sum_{n \geq 0} v_n x^n \mid v_n \in V, v_n = 0 \text{ for all but finitely many } n \right\}$$

(formal polynomials). Often our vector space V will be a vector space of endomorphisms, $\text{End } V$. Even when V is replaced by $\text{End } V$ some of these spaces are not algebras, and we must define multiplication only up to a natural restrictive condition, a *summability condition*. Let $f_i(x) = \sum_{r \in \mathbb{C}} a_i(r) x^r \in (\text{End } V)\{x\}$ for $1 \leq i \leq m$. Then the product

$$f_1(x)f_2(x) \cdots f_k(x)$$

exists if for every $m \in \mathbb{C}$ and $v \in V$

$$\sum_{n_1 + \cdots + n_k = m} a_1(n_1) \cdots a_k(n_k) v$$

is a finite sum. We will use routine extensions of this principle of summability without further comment. One must be careful with existence issues when dealing with associativity properties of such products. For instance, if $F(x), G(x)$ and $H(x) \in \text{End } V[[x, x^{-1}]]$ and if the three products $F(x)G(x)$, $G(x)H(x)$ and $F(x)G(x)H(x)$ all exist, then it is easy to verify that

$$(F(x)G(x))H(x) = F(x)(G(x)H(x)),$$

but unless some such condition is checked then “associativity” may fail. We shall use this type of associativity and routine extensions of the principle without further comment.

Remark 2.1. Throughout this paper, we often extend our spaces to include more than one variable. We state certain properties which have natural extensions in such multivariable settings, which we will also use without further comment.

We shall frequently use the notation e^w to refer to the formal exponential expansion, where w is any formal object for which such expansion makes sense (meaning that the coefficients are finitely computable). For instance, we have the linear operator $e^{y\frac{d}{dx}} : \mathbb{C}[[x, x^{-1}]] \rightarrow \mathbb{C}[[x, x^{-1}]][[y]]$:

$$e^{y\frac{d}{dx}} = \sum_{n \geq 0} \frac{y^n}{n!} \left(\frac{d}{dx} \right)^n.$$

Similiarly we let $\log(1 + w)$ refer to the formal logarithmic series

$$\log(1 + w) = \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} w^i,$$

whenever the expansion makes sense. We note that we shall also be using objects with names like $\log x$ which is a symbol for a new atomic object, not a shorthand for a series, but rather a single new formal variable, and it should not be confused with $\log(1 + x)$ which is a series in the entirely different formal variable, x .

Proposition 2.1. (*The “automorphism property”*) *Let A be an algebra over \mathbb{C} . Let D be a formal derivation on A . That is, D is a linear map from A to itself which satisfies the product rule:*

$$D(ab) = (Da)b + a(Db), \text{ for all } a \text{ and } b \text{ in } A.$$

Then

$$e^{yD}(ab) = (e^{yD}a) (e^{yD}b),$$

Proof. Notice that

$$D^n ab = \sum_{l+k=n}^r n! \frac{D^k a}{k!} \frac{D^l b}{l!}.$$

Then divide both sides by $n!$ and sum over y and the result follows. \square

Essentially for the same reason as the above proof (combinatorially speaking) we have for w_1 and w_2 commuting objects, the formal rule

$$(2.1) \quad e^{w_1+w_2} = e^{w_1} e^{w_2}.$$

More generally, we shall find the following formal rule very useful. If $[x, y]$ commutes with x and y then

$$(2.2) \quad e^x e^y = e^y e^x e^{[x, y]}.$$

We essentially follow the exposition for formula (3.4.7) in [FLM].

$$xe^y = x \sum_{n \geq 0} \frac{y^n}{n!} = \sum_{n \geq 0} \frac{y^n}{n!} x + \sum_{n \geq 1} \frac{ny^{n-1}}{n!} [x, y] = e^y x + e^y [x, y] = e^y (x + [x, y]),$$

and iterating gives

$$x^k e^y = e^y (x + [x, y])^k,$$

and dividing by $k!$ and summing over $k \geq 0$ gives

$$e^x e^y = e^y e^{x+[x,y]},$$

which gives the result because $[x, y]$ commutes with x .

We define

$$(2.3) \quad \binom{n}{k} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!}$$

for all $n \in \mathbb{C}$ and $k \in \mathbb{Z}$.

Definition 2.1.

$$(x + y)^r = e^{y \frac{d}{dx}} x^r \quad \text{for } r \in \mathbb{C}.$$

This definition is equivalent to the usual *binomial expansion convention*, the convention being that the second listed variable is expanded in nonnegative powers. The reader may easily check that this definition coincides with the usual definition of $(x + y)^n$ when n is a nonnegative integer. Other expansions, using (2.3), also “look” as expected. But we must be careful. The following nontrivial fact is true.

Proposition 2.2. *For all $n \in \mathbb{Z}$,*

$$(x + (y + z))^n = ((x + y) + z)^n.$$

Proof. Recalling (2.1), we have

$$(x + (y + z))^n = e^{(y+z) \frac{\partial}{\partial x}} x^n = e^{y \frac{\partial}{\partial x}} \left(e^{z \frac{\partial}{\partial x}} x^n \right) = e^{y \frac{\partial}{\partial x}} (x + z)^n = ((x + y) + z)^n.$$

□

Given formal commuting variables x, y etc., we take the point of view of certain elementary material in [M1] and [HLZ] and let $\log x, \log y$ etc. be formal variables commuting with x, y and each other etc. We further require that $\frac{d}{dx} \log x = x^{-1}$ and extend the operator to act as the unique possible derivation on $\mathbb{C}[x, \log x]$. We further “formally linearly complete” $\frac{d}{dx}$ to act on $\mathbb{C}[[x, x^{-1}, \log x, (\log x)^{-1}]]$ in the obvious way. We shall use other similar routine definitions without further comment.

Definition 2.2.

$$(\log(x + y))^r = e^{y \frac{d}{dx}} (\log x)^r \quad r \in \mathbb{C}.$$

It is easy to verify that

$$\log(x + y) = \log x + \log \left(1 + \frac{y}{x} \right).$$

The following is a special case of the first part of Theorem 3.6 in [HLZ].

Proposition 2.3. *(The logarithmic formal Taylor theorem)*

For $p(x) \in \mathbb{V}[[x, x^{-1}, \log x, (\log x)^{-1}]]$,

$$e^{y \frac{d}{dx}} p(x) = p(x + y).$$

Proof. A summand of $p(x)$ is doubly weighted by the degrees of the formal variables. The summands of $p(x + y)$ are doubly weighted by the total degree in x and y as well as separately in the degree in $\log x$. The operator preserves the total degree of the x and y variables. Thus we only need to consider $p(x)$ of the form $x^k q(x)$ where $k \in \mathbb{Z}$ and $q(x) \in \mathbb{V}[[\log x, (\log x)^{-1}]]$. Let $q_n(x)$ be the Laurent polynomial which is equal to $q(x)$ but truncated in powers whose absolute value is less than n . If we focus on a fixed power of y and separately of $\log x$ it is clear that we only need to consider $q_n(x)$ in place of $q(x)$. The result follows by linearity after considering the trivial cases $p(x) = x^r$ and $p(x) = (\log x)^r$ for $r \in \mathbb{C}$ and applying the automorphism property. \square

Proposition 2.4. *The vectors e^{mx} for $m \in \mathbb{C}$ are linearly independent.*

Proof. Otherwise there is a linear combination $\sum_{m \in \mathbb{C}} a_m e^{mx} = 0$, where $a_m \in \mathbb{C}$ and only finitely many coefficients are nonzero. Considering the higher derivatives and focusing on the constant term yields a Vandermonde matrix which is nonsingular which gives that the coefficients must all be zero. \square

We have therefore that the space spanned by the vectors $e^{m\xi_0}$ $m \in \mathbb{Z}$ is the space of Laurent polynomials $\mathbb{C}[e^{\xi_0}, e^{-\xi_0}]$. We tensor this space with the space of polynomials $\mathbb{C}[\xi_1, \xi_2, \dots]$ to get the space

$$(2.4) \quad \Xi = \mathbb{C}[e^{\xi_0}, e^{-\xi_0}, \xi_1, \xi_2, \dots].$$

Our vertex algebra will have Ξ as its underlying space. Of course, Ξ is also an algebra under the obvious rules.

3. THE DELTA FUNCTION

We define the formal delta function by

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n.$$

Certain elementary identities concerning delta functions are very convenient for dealing with the arithmetic of vertex algebras and, in fact, in some cases, are fundamental to the very notion of vertex algebra. We state and prove some such identities in this section. The following identity appeared in [FLM]. Our proof follows that given in [R].

Proposition 3.1. *We have the following two elementary identities:*

$$(3.1) \quad y^{-1} \delta \left(\frac{x - z}{y} \right) - x^{-1} \delta \left(\frac{y + z}{x} \right) = 0$$

and

$$(3.2) \quad z^{-1}\delta\left(\frac{x-y}{z}\right) - z^{-1}\delta\left(\frac{-y+x}{z}\right) - x^{-1}\delta\left(\frac{y+z}{x}\right) = 0.$$

Proof. First observe that

$$(3.3) \quad y^{-1}\delta\left(\frac{x}{y}\right) = \sum_{l<0} x^l y^{-l-1} + \sum_{l\geq 0} x^l y^{-l-1} = e^{-y\frac{d}{dx}} x^{-1} + e^{-x\frac{d}{dy}} y^{-1} = (x-y)^{-1} + (y-x)^{-1}.$$

Then by the formal Taylor theorem

$$\begin{aligned} y^{-1}\delta\left(\frac{x+z}{y}\right) &= e^{z\frac{d}{dx}} y^{-1}\delta\left(\frac{x}{y}\right) \\ &= e^{z\frac{d}{dx}} ((x-y)^{-1} + (y-x)^{-1}) \\ &= ((x+z)-y)^{-1} + (y-(x+z))^{-1}. \end{aligned}$$

Being careful with minus signs, we may respectively expand all the terms in the left-hand side of (3.2) to get

$$\begin{aligned} &((x-y)-z)^{-1} + (z-(x-y))^{-1} \\ &-((-y+x)-z)^{-1} - (z-(-y+x))^{-1} \\ &-((y+z)-x)^{-1} - (x-(y+z))^{-1}. \end{aligned}$$

Now we get by Proposition 2.2 that the first and sixth terms, the third and fifth terms, and the second and fourth terms pairwise cancel each other thus giving us (3.2). The other identity may be proved in a similar fashion. \square

We note the following easily verified identity extending (3.3):

$$(3.4) \quad \frac{1}{n!} \left(\frac{\partial}{\partial y}\right)^n y^{-1}\delta\left(\frac{x}{y}\right) = \frac{(-1)^n}{n!} \left(\frac{\partial}{\partial x}\right)^n y^{-1}\delta\left(\frac{x}{y}\right) = (x-y)^{-n-1} - (-y+x)^{-n-1}.$$

In addition to the above identities, delta functions have certain crucial substitution properties. To state such a property in the generality needed, we need the following logarithmic substitution operator. It is crucial to note as in [HLZ] that whereas

$$\log(e^x) = \log(1 + (e^x - 1)) = x$$

it is *not* true that

$$e^{\log x} = x,$$

since the left hand side is a series in the variable $\log x$. But nonetheless, the two expressions are heuristically equal and we develop a means of rigorizing this heuristic suggestion next. For V which does not depend on $\log x$ or x we shall define an operator

$$\phi_x : V[[x, x^{-1}]] [e^{\log x}, e^{-\log x}] \rightarrow V[[x, x^{-1}]].$$

The operator ϕ_x essentially substitutes x for $e^{\log x}$. For $f(x) = \sum_{n \in \mathbb{Z}} v_n(x) e^{n \log x} \in V[[x, x^{-1}]] [e^{\log x}, e^{-\log x}]$ where $v_n(x) \in V[[x, x^{-1}]]$, let

$$\phi_x(f(x)) = \sum_{n \geq 0} v_n(x) x^n.$$

This is well defined because $v_n(x) x^n$ is a well defined formal Laurent series and $\phi_x(f(x))$ is a finite sum of such series. We shall use the notation

$$\phi_{x,y} = \phi_x \circ \phi_y$$

etc., if we wish to indicate the operator substituting for more than one logarithmic variable in a similar way.

Proposition 3.2. *The operator ϕ_x commutes with $\frac{d}{dx}$.*

Proof. It is enough to consider acting on elements of the form $x^n e^{m \log x}$. We have

$$\phi_x \frac{d}{dx} x^n e^{m \log x} = \phi_x ((n+m)x^{n-1} e^{m \log x}) = (m+n)x^{m+n-1}$$

and

$$\frac{d}{dx} \phi_x x^n e^{m \log x} = \frac{d}{dx} x^{n+m} = (m+n)x^{m+n-1}.$$

□

The following is an example of how we shall typically be applying ϕ_x ,

$$\phi_x e^{-2 \log(x-z)} = \phi_x e^{-z \frac{\partial}{\partial x}} e^{-2 \log x} = e^{-z \frac{\partial}{\partial x}} \phi_x e^{-2 \log x} = e^{-z \frac{\partial}{\partial x}} x^{-2} = (x-z)^{-2}.$$

Proposition 3.3. *For*

$$f(x, y, z) \in \text{End } V[[x, x^{-1}, e^{\log x}, e^{-\log x}, y, y^{-1}, e^{\log y}, e^{-\log y}, z, z^{-1}, e^{\log z}, e^{-\log z}]]$$

such that for each fixed $v \in V$

$$f(x, y, z)v \in \text{End } V[[x, x^{-1}, y, y^{-1}]] [e^{\log x}, e^{-\log x}, e^{\log y}, e^{-\log y}]((z)) [e^{\log z}, e^{-\log z}]$$

and such that

$$\lim_{x \rightarrow y} f(x, y, z)$$

exists (where the “limit” is the indicated formal substitution), we have

$$\begin{aligned} \phi_{x,y,z} \delta \left(\frac{y+z}{x} \right) f(x, y, z) &= \phi_{x,y,z} \delta \left(\frac{y+z}{x} \right) f(y+z, y, z) \\ &= \phi_{x,y,z} \delta \left(\frac{y+z}{x} \right) f(x, x-z, z). \end{aligned}$$

Proof. Since we are substituting terms homogeneous of degree 1, for other terms homogeneous of degree 1, therefore after $\phi_{x,y,z}$ is applied we need only consider homogeneous $f(x, y, z)$. The limit restriction on $f(x, y, z)$ together with the fact that the logarithmic terms only involve finite sums means that we need only consider when

$$f(x, y, z) = x^{l_1} e^{l_2 \log x} y^{m_1} e^{m_2 \log y} z^{n_1} e^{n_2 \log z}.$$

This follows from an easy calculation. \square

4. SOME GENERAL LOG VERTEX OPERATORS

Let $x, \log x$ and ξ_i for $i \geq 0$ be formal commuting variables and consider the space $\mathbb{C}[\xi_1, \xi_2, \dots][[\xi_0]]$. We define some operators on this space (cf. (4.2.1) in [FLM]). Let

$$h^+(x) = 2 \log x \frac{\partial}{\partial \xi_0} - 2 \sum_{n>0} \frac{x^{-n}}{n} \frac{\partial}{\partial \xi_n}$$

and

$$h^-(x) = \sum_{m \geq 0} \xi_m x^m.$$

Let

$$h(x) = h^+(x) + h^-(x),$$

so that

$$h(x) : \mathbb{C}[\xi_1, \xi_2, \dots][[\xi_0]] \rightarrow \mathbb{C}[\xi_1, \xi_2, \dots][[\xi_0]]((x))[\log x].$$

We use colons to denote normally ordered products, where normal ordering places differential operators to the right side of products. For instance, for $n \geq 0$

$$: \frac{\partial}{\partial \xi_n} \xi_n := \xi_n \frac{\partial}{\partial \xi_n} := \xi_n \frac{\partial}{\partial \xi_n},$$

with the notation extended in the obvious way over longer products. We have

$$: h^-(x) h^+(x) := h^-(x) h^+(x) := h^-(x) h^+(x),$$

so that for $m \in \mathbb{Z}$ and $n \geq 0$

$$(4.5) \quad : mh(x)^n := (mh^-(x) + mh^+(x))^n := \sum_{l+k=n} \frac{(mh^-(x))^l}{l!} \frac{(mh^+(x))^k}{k!}.$$

Let us consider $: mh(x)^n :$ acting on an element of $\mathbb{C}[\xi_1, \xi_2, \dots][[\xi_0]]$. In particular, focusing on the coefficient of x and separately $\log x$ and separately ξ_0 all raised to fixed powers then it is clear that the answer is zero for sufficiently large n . Therefore dividing by $n!$ and summing, we get the well defined operator (cf. (4.2.6) and also (4.2.12) in [FLM])

$$: e^{mh(x)} :.$$

Recall (2.4), the space

$$\Xi = \mathbb{C}[e^{\xi_0}, e^{-\xi_0}, \xi_1, \xi_2, \dots].$$

We may consider $: e^{mh(x)} :$ acting on Ξ as in the next proposition.

Proposition 4.1. *For $m \in \mathbb{Z} : e^{mh(x)} :: \Xi \rightarrow \Xi((x))[e^{\log x}, e^{-\log x}]$*

$$: e^{mh(x)} := e^{mh^-(x)} e^{mh^+(x)} = e^{m\xi_0} e^{mh^-(x)-m\xi_0} e^{mh^+(x)-2\log x \frac{\partial}{\partial \xi_0}} e^{2\log x \frac{\partial}{\partial \xi_0}}.$$

Proof. Divide (4.5) by $n!$ and sum over $n \geq 0$. That we may restrict the domain and range follows easily from the expanded formula. \square

We define a linear and “normal multiplicative” map on Ξ . We shall explain “normal multiplicative” in a moment. First let

$$(4.6) \quad \Upsilon(1, x) = 1.$$

Next, for $n \geq 1$, let

$$\Upsilon(\xi_n, x) = \frac{h^{(n)}(x)}{n!}.$$

And extend naturally to let

$$(4.7) \quad \Upsilon(e^{m\xi_0}, x) =: e^{mh(x)} :,$$

for $m \in \mathbb{Z}$.

Remark 4.1. The reader should compare the operator (4.7) with the heuristic operator in Remark 4.2.1 of [FLM], which provided the original motivation for developing this, now rigorous, operator.

Then for $p, q \in \Xi$ we require that

$$(4.8) \quad \Upsilon(pq, x) =: \Upsilon(p, x)\Upsilon(q, x) :,$$

(which is what we mean by “normal multiplicative”). This determines a unique linear map

$$\Upsilon(\cdot, x) : \Xi \otimes \Xi \rightarrow \Xi((x))[e^{\log x}, e^{-\log x}].$$

Extending the notation in the obvious way, we have

$$\Upsilon(h^-(z), x) = \Upsilon\left(\sum_{m \geq 0} \xi_m z^m, x\right) = \sum_{m \geq 0} \frac{h^{(m)}(x)}{m!} z^m = e^{z \frac{d}{dx}} h(x) = h(x + z).$$

So

$$(4.9) \quad \Upsilon(e^{h^-(z)}, x) =: e^{h(x+z)} :,$$

and the range of this operator is $\Xi((x))[e^{\log x}, e^{-\log x}][[z]]$, which is easy to see using Proposition 2.3. More generally, the range of

$$: e^{h(x+y)} e^{h(x+z)} :$$

is $\Xi((x))[e^{\log x}, e^{-\log x}][[y, z]]$ which can be seen by first considering

$$: e^{h(x_1+y)} e^{h(x_2+z)} :$$

and then setting x_1 and x_2 equal to x . We shall use such reasoning without comment below.

We have

$$\begin{aligned} h^+(x) e^{nh^-(y)} &= \left(2 \log x \frac{\partial}{\partial \xi_0} - 2 \sum_{k>0} \frac{x^{-k}}{k} \frac{\partial}{\partial \xi_k} \right) e^{n \sum_{m \geq 0} \xi_m y^m} \\ &= \left(2n \log x - 2 \sum_{k>0} \frac{n}{k} y^k x^{-k} \right) e^{nh^-(y)} \\ &= 2n \left(\log x + \log \left(1 - \frac{y}{x} \right) \right) e^{nh^-(y)} \\ &= 2n \log(x - y) e^{nh^-(y)}, \end{aligned}$$

so that, letting $k = 2mn$,

$$(4.10) \quad e^{-mh^+(x)} e^{nh^-(y)} = e^{-k \log(x-y)} e^{nh^-(y)}.$$

We note that we may set $y = 0$ in this formula. We will use this later.

5. JACOBI IDENTITY

For $m, n \in \mathbb{Z}$, recalling (2.2)

$$: e^{mh(x)} :: e^{nh(z)} : = e^{mh^-(x)} e^{mh^+(x)} e^{nh^-(z)} e^{nh^+(z)} =: e^{mh(x)+nh(z)} : e^{[mh^+(x), nh^-(z)]},$$

because (cf. the proof of Proposition 4.3.1 in [FLM])

$$\begin{aligned} [h^+(x), h^-(z)] &= \left[2 \frac{\partial}{\partial \xi_0} \log x - 2 \sum_{n>0} \frac{1}{n} \frac{\partial}{\partial \xi_n} x^{-n}, \sum_{m \geq 0} \xi_m z^m \right] \\ &= \left[2 \frac{\partial}{\partial \xi_0}, \xi_0 \right] \log x + \sum_{n, m > 0} \left[\frac{-2}{n} \frac{\partial}{\partial \xi_n}, \xi_m \right] x^{-n} z^m \\ &= 2 \log x - 2 \sum_{n>0} \frac{1}{n} \left(\frac{z}{x} \right)^n \\ &= 2 \log x + 2 \log \left(1 - \frac{z}{x} \right) \\ &= 2 \log(x - z) \end{aligned}$$

is central, and moreover we have

$$(5.11) \quad : e^{mh(x)} :: e^{nh(z)} : =: e^{mh(x)+nh(z)} : e^{2mn \log(x-z)}.$$

The reader should compare the calculation here with the work in Section 8.6 of [FLM]. Let x_i, y_i, z_i for $i \geq 1$ and for convenience we also let $x_0 = y_0 = z_0 = 0$. Then let

$$A = e^{\sum_{i \geq 1} -m_i h^-(x_i)} e^{-m_0 \xi_0}$$

and

$$B = e^{\sum_{j \geq 1} n_j h^-(y_j)} e^{n_0 \xi_0},$$

where $m_i, n_j \in \mathbb{Z}$ and only finitely many are nonzero (cf. (8.6.5) in [FLM]). Then letting $k_{ij} = 2m_i n_j$,

$$\begin{aligned} \phi_{x,y,z} \Upsilon(A, x) \Upsilon(B, y) &= \phi_{x,y,z} : e^{\sum_{i \geq 0} -m_i h(x+x_i)} :: e^{\sum_{j \geq 0} n_j h(y+y_j)} : \\ &= \phi_{x,y,z} : e^{\sum_{i \geq 0} -m_i h(x+x_i) + \sum_{j \geq 0} n_j h(y+y_j)} : e^{\sum_{i,j \geq 0} -k_{ij} \log(x+x_i-y-y_j)} \\ &= \phi_{x,y,z} : e^{\sum_{i \geq 0} -m_i h(x+x_i) + \sum_{j \geq 0} n_j h(y+y_j)} : \prod_{i,j \geq 0} (x+x_i-y-y_j)^{-k_{ij}}, \end{aligned}$$

by (4.8), (4.9) and (5.11) and its range is $\Xi((x))[[x_1, x_2, \dots]]((y))[[y_1, y_2, \dots]]$. Therefore we may multiply this expression by $z^{-1} \delta\left(\frac{x-y}{z}\right)$ and Proposition 3.3 may be applied to give

$$\begin{aligned} \phi_{x,y,z} z^{-1} \delta\left(\frac{x-y}{z}\right) \Upsilon(A, x) \Upsilon(B, y) \\ (5.12) \quad = \phi_{x,y,z} z^{-1} \delta\left(\frac{x-y}{z}\right) : e^{\sum_{i \geq 0} -m_i h(x+x_i) + \sum_{j \geq 0} n_j h(y+y_j)} : \prod_{i,j \geq 0} (z+x_i-y_j)^{-k_{ij}}. \end{aligned}$$

Similarly

$$\begin{aligned} \phi_{x,y,z} z^{-1} \delta\left(\frac{y-x}{-z}\right) \Upsilon(B, y) \Upsilon(A, x) \\ (5.13) \quad = \phi_{x,y,z} z^{-1} \delta\left(\frac{y-x}{-z}\right) : e^{\sum_{i \geq 0} -m_i h(x+x_i) + \sum_{j \geq 0} n_j h(y+y_j)} : \prod_{i,j \geq 0} (z+x_i-y_j)^{-k_{ij}}, \end{aligned}$$

noting that $(-z-x_i+y_j)^{-k_{ij}} = (z+x_i-y_j)^{-k_{ij}}$, since k_{ij} is always even.

Further

$$\begin{aligned} \Upsilon(\Upsilon(A, z) B, y) &= \Upsilon\left(: e^{\sum_{i \geq 0} -m_i h(z+x_i)} : e^{\sum_{j \geq 1} n_j h^-(y_j) + n_0 \xi_0}, y\right) \\ &= e^{\sum_{i,j \geq 0} -k_{ij} \log(z+x_i-y_j)} \Upsilon\left(e^{\sum_{i \geq 0} -m_i h^-(z+x_i) + \sum_{j \geq 1} n_j h^-(y_j) + n_0 \xi_0}, y\right) \\ &= e^{\sum_{i,j \geq 0} -k_{ij} \log(z+x_i-y_j)} : e^{\sum_{i \geq 0} -m_i h(y+z+x_i) + \sum_{j \geq 0} n_j h(y+y_j)} :, \end{aligned}$$

by (4.8), (4.9) and (4.10), and its range is

$$\Xi((y))[e^{\log y}, e^{-\log y}][[y_1, y_2, \dots]]((z))[e^{\log z}, e^{-\log z}][[x_1, x_2, \dots]],$$

so that we may multiply by $y^{-1}\delta\left(\frac{x-z}{y}\right)$ and Proposition 3.3 applies, giving

$$\begin{aligned}
 & \phi_{x,y,z} x^{-1} \delta\left(\frac{y+z}{x}\right) \Upsilon(\Upsilon(A, z) B, y) \\
 (5.14) \quad &= \phi_{x,y,z} x^{-1} \delta\left(\frac{y+z}{x}\right) \prod_{i,j \geq 0} (z + x_i - y_j)^{-k_{ij}} : e^{\sum_{i \geq 0} -m_i h(x+x_i) + \sum_{j \geq 0} n_j h(y+y_j)} : .
 \end{aligned}$$

Now adding the left hand sides of (5.12), (5.13) and (5.14) together, factoring and using Proposition 3.1, we get:

$$\begin{aligned}
 & \phi_{x,y,z} z^{-1} \delta\left(\frac{x-y}{z}\right) \Upsilon(A, x) \Upsilon(B, y) - \phi_{x,y,z} z^{-1} \delta\left(\frac{y-x}{-z}\right) \Upsilon(B, y) \Upsilon(A, x) \\
 (5.15) \quad &= \phi_{x,y,z} x^{-1} \delta\left(\frac{y+z}{x}\right) \Upsilon(\Upsilon(A, z) B, y) .
 \end{aligned}$$

6. A VERTEX ALGEBRA

The operator $\Upsilon(\cdot, x)$ is almost a vertex operator, but we need to specialize $e^{\log x}$ to x . So we let

$$(6.16) \quad Y(\cdot, x) \cdot = \phi_x \circ \Upsilon(\cdot, x) \cdot$$

so that

$$Y(\cdot, x) \cdot : \Xi \otimes \Xi \rightarrow \Xi((x))$$

is a linear map.

We are now ready to recall the definition of a vertex algebra. Individual mathematical vertex operators were introduced in [LW]. The notion of vertex algebra was first mathematically defined by Borchers in [B]. An equivalent set of axioms, based primarily on the Jacobi identity, which we shall use, appeared first in [FLM] as part of the notion of vertex operator algebra (cf. [LL], in which the equivalence of the two sets of axioms is proved).

Definition 6.1. A *vertex algebra* is a vector space equipped, first, with a linear map (the *vertex operator map*) $V \otimes V \rightarrow V[[x, x^{-1}]]$, or equivalently, a linear map

$$\begin{aligned}
 Y(\cdot, x) : V &\rightarrow (\text{End} V)[[x, x^{-1}]] \\
 v &\mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} .
 \end{aligned}$$

We call $Y(v, x)$ the *vertex operator associated with* v . We assume that

$$Y(u, x)v \in V((x))$$

for all $u, v \in V$. There is also a distinguished element $\mathbf{1}$ satisfying the following *vacuum property* :

$$Y(\mathbf{1}, x) = 1$$

and *creation property* :

$$\begin{aligned} Y(u, x)\mathbf{1} &\in V[[x]] \text{ and} \\ Y(u, 0)\mathbf{1} &= u \text{ for all } u \in V. \end{aligned}$$

Finally, we require that the *Jacobi identity* is satisfied:

$$\begin{aligned} z^{-1}\delta\left(\frac{x-y}{z}\right)Y(u, x)Y(v, y) - z^{-1}\delta\left(\frac{y-x}{-z}\right)Y(v, y)Y(u, x) \\ = x^{-1}\delta\left(\frac{y+z}{x}\right)Y(Y(u, z)v, y). \end{aligned}$$

Theorem 6.1. *The space Ξ equipped with $Y(\cdot, x)\cdot$ as defined by (6.16), is a vertex algebra, with vacuum vector $\mathbf{1}$.*

Proof. The minor conditions, such as truncation have already been dealt with. The vacuum property follows by (4.6). To see the creation property, note that

$$Y(A, x)\mathbf{1} = \phi_x : e^{\sum_{i \geq 0} -m_i h(x+x_i)} : \mathbf{1} = e^{\sum_{i \geq 0} -m_i h^-(x+x_i)},$$

which is a power series in x and we may substitute $x = 0$ which is easily seen to give

$$Y(A, 0)\mathbf{1} = A.$$

Noting (5.15), we will be done if we can show that the coefficients in A and B span Ξ as we let their parameters vary.

The following is essentially the same observation as in Remark 8.3.9 in [FLM]. Let

$$e^{\sum_{n \geq 1} \xi_n y^n} = \sum_{k \geq 0} p_k y^k \quad \text{so that} \quad \sum_{n \geq 1} \xi_n y^n = \log \left(1 + \sum_{k \geq 1} p_k y^k \right),$$

which shows that any polynomial in the ξ_i $i \geq 1$ is a polynomial in the p_i for $i \geq 1$. This gives the result. \square

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